UNBOUNDED SOLUTIONS OF A SECOND-ORDER DIFFERENTIAL EQUATION
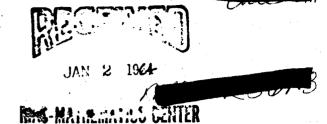
WITH NON-NEGATIVE DAMPING

by

G. R. Morris*

1. Introduction

It is plausible that, if $f(x,y) \geq 0$ for all x and y, every solution of

$$\ddot{x} + f(x,\dot{x})\dot{x} + g(x) = p(t) \tag{1.1}$$

will be bounded, this suggestion being subject to the proviso that the equation $\ddot{x} + g(x) = p(t)$ should be (in some sense) clearly different from a linear differential equation exhibiting resonance. We might indeed expect that if $f(x,y) > 0$ in a considerable part of the plane every solution of (1.1) will eventually satisfy $|x(t)| + |\dot{x}(t)| < B$ where B is an absolute constant.
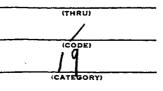
In this note I construct an equation of the form (1.1), as near as desired to the equation

$$\ddot{x} + x = -8 \sin 3t, \tag{1.2}$$

which has a one-parameter family of unbounded solutions.

<u>Proof of Theorem 2</u>. Suppose $x(t)$ is the solution of (4.1) with $\dot{x}(0) = 3$ and $x(0) = A$, A being large, and write $B = x(\beta)$, where β is the smallest positive t making $\dot{x}(t) = 0$. Write $x'(t)$ for the solution of (4.3) with $x'(\beta) = B$, $\dot{x}'(\beta) = 0$ and write $t(y, \beta, B)$ and $t'(y, \beta, B)$ for the functions inverse to $y(t) = \dot{x}(t)$ and $y'(t) = \dot{x}'(t)$ respectively. Since, evidently, $B = A + \underline{O}(1)$ and $\beta = \underline{O}(A^{-1})$ the hypotheses of Lemma 8 are satisfied. We have therefore

$$0 = t(3, \beta, B) > t'(3, \beta, B)$$

except when (4.5) holds, that is, in terms of the original functions, $\dot{x}'(0) < 3$ except when $f(x(t), \dot{x}(t)) = 0$ for $0 \leq t \leq \beta$.

In the exceptional case $x(t) = x'(t)$ for all t, and hence $f(x(t), x(t)) = 0$ for $0 \leq t \leq \beta$ if and only if this holds for all t.

When $t'(3, \beta, B) < 0$, write $-\delta$ for its value. Evidently $x'(t)$ has period 2π, and, since for $\beta \leq t \leq 2\pi - \delta$ we have $x(t) = x'(t)$, we obtain

$$\dot{x}(2\pi - \delta) = \dot{x}'(2\pi - \delta) = \dot{x}'(-\delta) = 3,$$

which at once gives $\dot{x}(2\pi) < 3$.

I should like to thank Dr. J. K. Hale and Mr. C. Perelló for their comments on this paper.

RIAS, Baltimore. (After March 1964: Department of Mathematics, University of Queensland, Brisbane, Australia)

THEOREM 1.   It is possible so to define the continuous functions f(x,y), positive in the half-strip $x > x_c^*$, $0 < y < 3$ and zero elsewhere, and g(x), equal to x in $|x| \geq 1$, that every solution of

$$\ddot{x} + f(x,\dot{x})\dot{x} + g(x) = -8 \sin 3t \qquad (1.3)$$

with x(0) large and $\dot{x}(0) = 3$ has $|x(t)| + |\dot{x}(t)| \to \infty$. Further, for every small positive ϵ it is possible to define such functions f(x,y) and g(x) with the additional properties that

$$0 \leq f(x,y) < \epsilon \quad \text{and} \quad |x - g(x)| < \epsilon$$

for all x,y.

We shall proceed by modifying (1.2). Clearly any solution of this linear equation can be written as $x = R \cos (t - \beta) + \sin 3t$, which is to say that any solution of the system
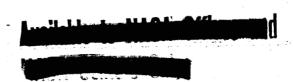
$$\dot{x} = y, \quad \dot{y} = -x - 8 \sin 3t$$

can be written

$$x = R \cos (t - \beta) + \sin 3t,$$

$$y = -R \sin (t - \beta) + 3 \cos 3t.$$

Having this explicit form for solution curves in (x, y, t)-space, we can easily (and in more than one way) determine tubes on which they lie; we shall

need the particular result that any solution curve remains on a tube

$$(x - \sin 3t)^2 + (y - 3 \cos 3t)^2 = R^2 . \qquad (1.4)$$

Although it is tempting to say that (in whatever sense may be relevant to our work) such a tube will be "near enough" to a circular cylinder having the t-axis as axis of symmetry, it will appear from our calculations that the directions of normals to tubes and cylinders are not near enough for us to deduce the boundedness of solutions of (1.1). This clash between a plausible guess and exact knowledge may serve to explain in a geometrical way why the plausible suggestion for (1.1) is false.

The unit inward normal to the tube (1.4) is a positive multiple of

$$\underset{\sim}{N} = (-(x - \sin 3t), -(y - 3 \cos 3t), 3x \cos 3t - 9y \sin 3t + 12 \sin 6t),$$

and we can assert that a vector $\underset{\sim}{v}$ at (x,y,t) points out of the tube if $\underset{\sim}{N} \cdot \underset{\sim}{v} < 0$. In particular the vector $(\dot{x},\dot{y},1)$ corresponding to the modified system

$$\dot{x} = y, \; \dot{y} = -x - fy - 8 \sin 3t \qquad (1.5)$$

points out of the tube when

$$fy \, (y - 3 \cos 3t) < 0.$$

When $t = 0$, 2π, 4π, ... this condition is satisfied in $0 < y < 3$
for any choice of positive f. This suggests that, for some positive x*,
we should try to determine $f(x,y)$, positive in $x > x*$, $0 < y < 3$, in
such a way that any solution curve of (1.5) which had x large and
$y = 3$ for $t = 0$ would next have $y = 3$ and x positive when $t = 2\pi$.
However, as we shall show in §4 (as Theorem 2), such a determination is not
possible; we, therefore, arrange that $y = 3$ successively at $t = 0$,
2π, 4π, ... by changing x into a cubic polynomial for $|x| \leq 1$, a pro-
cedure which unfortunately leads us into some calculations.

2. Estimates for special equations

We shall first show how to determine an $h(x,y)$ having all the
properties asserted of f except continuity on the boundaries of the half-strip.
Since the signs of coefficients are important in the discussion we need to
give the calculations in some detail; it will be convenient to set them out
as a string of lemmas concerning the equation (1.2) and its two perturbations

$$\ddot{x} + \lambda\dot{x} + x = -8 \sin 3t \tag{2.1}$$

and

$$\ddot{x} + x + \mu(x^3 - x) = -8 \sin 3t, \tag{2.2}$$

where λ and μ are parameters. It is to be observed that although they have
notation in common, the lemmas are independent of one another. Finally we

show that our $h(x,y)$ can be modified to give a continuous $f(x,y)$.

All our variables and parameters are real. We shall specify the largeness of our solutions by using parameters A_i, and shall allow ourselves the license of saying that a function which admits an expansion

$$A_i + \sum_{n=0}^{\infty} c_n A_i^{-n},$$

convergent for $A_i > \alpha$, is <u>analytic in</u> A_i^{-1} <u>for large</u> A_i. We shall consistently assert the analytic dependence of the solutions of our differential equations on the parameters and initial values. We make no deep use of this analyticity, and indeed could neglect it at the price of calculating additional error terms; we shall exploit it in the proof of Theorem 1 so as to speak unambiguously of "the terms free of λ and μ" in $\varphi_6(A,\lambda,\mu)$.

Although the geometrical point of view motivates our construction we shall not appeal to it explicitly in the calculations but instead estimate the change of the non-negative function $R^2(x,y,t)$ defined by (1.4). When we are working with a function $x(t)$, we shall abbreviate $R^2\{x(t), y(t), t\}$ to $R^2(t)$ without special mention.

LEMMA 1. <u>Suppose</u> A_o <u>is large and</u> $|\lambda|$ <u>small, and let</u> $\varphi_1 = \varphi_1(A_o,\lambda)$ <u>be the least positive</u> t <u>such that the solution of</u> (2.1) <u>with</u> $x(0) = A_o$, $\dot{x}(0) = 3$ <u>has</u> $\dot{x}(t) = 0$. <u>Then</u> φ_1 <u>and</u> $x(\varphi_1)$ <u>are analytic in</u>

A_o^{-1} and λ,

$$\varphi_1 = \underline{0}\,(A_o^{-1}),$$

$$x(\varphi_1) = A_o + \underline{0}\,(A_o^{-1}),$$

$$\varphi_1(A_o,\lambda) = \varphi_1(A,0) - \frac{9}{2}\lambda A_o^{-2} + \underline{0}\,(|\lambda|A_o^{-3}) \qquad (2.3)$$

and

$$R^2(\varphi_1) - R^2(0) = 9\lambda A_o^{-1}\{1 + \underline{0}(A_o^{-1})\}\,. \qquad (2.4)$$

Replace (2.1) by the system

$$\dot{x} = y, \quad \dot{y} = -x - \lambda y - 8 \sin 3t,$$

write $x = A_o z$ and change to y as independent variable, obtaining

$$\frac{dt}{dy} = -\frac{A_o^{-1}}{z + \lambda A_o^{-1}y + 8\,A_o^{-1}\sin 3t}\,,$$

$$\frac{dz}{dy} = -\frac{A_o^{-2}y}{z + \lambda A_o^{-1}y + 8\,A_o^{-1}\sin 3t}\,.$$

The analytic dependence on A_o^{-1} and λ of the solution having $t = 0$ and $z = 1$ for $y = 3$ is now clear. Further,

$$\frac{\partial}{\partial y}\left[\frac{\partial t}{\partial \lambda}\right] = A_o^{-1} \cdot \frac{\partial z/\partial \lambda + A_o^{-1}y + 24\,A_o^{-1}\cos 3t \cdot \partial t/\partial \lambda}{(z + \lambda A_o^{-1}y + 8\,A_o^{-1}\sin 3t)^2}\,. \qquad (2.5)$$

If the maximum of $|\partial t/\partial \lambda|$ in $0 \leq y \leq 3$ is attained where $y = y^*$, we see, by integrating between 3 and y^*, that

$$\max\left| \frac{\partial t}{\partial \lambda} \right| \leq 2A_o^{-1}\left\{ \max\left| \frac{\partial z}{\partial \lambda} \right| + 3A_o^{-1} + 24\,A_o^{-1}\max\left| \frac{\partial t}{\partial \lambda} \right| \right\}$$

$$= 2A_o^{-1}\,M, \quad \text{say.}$$

Since, similarly, $\max|\partial z/\partial \lambda| \leq 6A_o^{-2}\,M$ we obtain

$$M \leq 6A_o^{-2}M + 3A_o^{-1} + 48A_o^{-2}M,$$

which gives $M = \underline{O}\,(A_o^{-1})$ and then $\partial t/\partial \lambda = \underline{O}\,(A_o^{-2})$, $\partial z/\partial \lambda = \underline{O}\,(A_o^{-3})$. We may now deduce from (2.5) that

$$\frac{\partial \varphi_1}{\partial \lambda} = \int_3^o \frac{\partial}{\partial y}\left(\frac{\partial t}{\partial \lambda}\right) dy = -\int_o^3 A_o^{-2}y\,\{1 + \underline{O}\,(A_o^{-1})\}dy$$

$$= -\frac{9}{2}\,A_o^{-2}\,\{1 + \underline{O}\,(A_o^{-1})\},$$

which implies (2.3). The estimate of $x(\varphi_1)$ is immediate and since

$$\frac{d}{dy}\,(R^2) = \frac{2\lambda A_o^{-1}y(y - 3\cos 3t)}{z + \lambda A_o^{-1}y + 8\,A_o^{-1}\sin 3t}$$

$$= 2\lambda A_o^{-1}y\,\{\,y - 3 + \underline{O}\,(A_o^{-2})\}\,\{1 + \underline{O}\,(A_o^{-1})\}$$

$$= 2\lambda A_o^{-1}(y^2 - 3y) + \underline{O}\,(\lambda A_o^{-2}),$$

an integration gives $R^2(\varphi_1) - R^2(0)$.

LEMMA 2. $\underline{\text{Suppose}}$ A_1 $\underline{\text{is large and let}}$ φ_2 $\underline{\text{be the least}}$ t $\underline{\text{exceeding}}$ φ_1 $\underline{\text{such that the solution of}}$ (1.2) $\underline{\text{with}}$ $x(\varphi_1) = A_1$, $\dot{x}(\varphi_1) = 0$ $\underline{\text{has}}$ $x(t) = 1$. $\underline{\text{Then}}$ φ_2 $\underline{\text{and}}$ $\dot{x}(\varphi_2)$ $\underline{\text{are analytic in}}$ A_1^{-1} $\underline{\text{and}}$ φ_1,

$$\varphi_2 - \varphi_1 = \tfrac{1}{2}\pi + \underline{O}\,(A_1^{-1})$$

$\underline{\text{and}}$

$$\dot{x}(\varphi_2) = -A_1 + \underline{O}(1) \ .$$

LEMMA 3. $\underline{\text{Suppose}}$ A_2 $\underline{\text{is large and}}$ $|\mu|$ $\underline{\text{small, and let}}$ $\varphi_3 = \varphi_3(A_2,\ \mu,\ \varphi_2)$ $\underline{\text{be the least}}$ t $\underline{\text{exceeding}}$ φ_2 $\underline{\text{such that the solution}}$ $\underline{\text{of}}$ (2.2) $\underline{\text{with}}$ $x(\varphi_2) = 1$, $\dot{x}(\varphi_2) = -A_2$ $\underline{\text{has}}$ $x(t) = -1$. $\underline{\text{Then}}$ φ_3 $\underline{\text{and}}$ $\dot{x}(\varphi_3)$ $\underline{\text{are analytic in}}$ A_2^{-1}, μ $\underline{\text{and}}$ φ_3,

$$\varphi_3 = \varphi_2 + \underline{O}\,(A_2^{-1}) \ ,$$

$$\dot{x}(\varphi_3) = -A_2 + \underline{O}\,(A_2^{-1}),$$

$$\varphi_3(A_2,\ \mu,\ \varphi_2) = \varphi_3(A_2,\ 0,\ \varphi_2) + \tfrac{4}{15}\,\mu\,A_2^{-3} + \underline{O}\,(|\mu|A_2^{-5}) \tag{2.6}$$

$\underline{\text{and}}$

$$R^2(\varphi_3) - R^2(\varphi_2) = -\tfrac{24}{5}\,\mu\,\sin 3\varphi_2 A_2^{-2} + \underline{O}\,(|\mu|A_2^{-3}). \tag{2.7}$$

In the interval $-1 \leqq x \leqq 1$ we have the approximate first integral

$$\dot{x}^2 + (1 - \mu)x^2 + \tfrac{1}{2}\mu x^4 = A_2^2 + \underline{O}\,(1)$$

whence we obtain our estimates for φ_3 and $\dot{x}(\varphi_3)$.

The asserted analytic behaviour is evident when we write $\dot{x} = y = -A_2 w$ and replace the given equation by the system

$$\frac{dt}{dx} = - \frac{1}{A_2 w} \quad ,$$

$$\frac{dw}{dx} = - A_2^{-2} w^{-1} \{x + \mu(x^3 - x) + 8 \sin 3t\} \quad .$$

We have further

$$\frac{\partial}{\partial x} \left(\frac{\partial t}{\partial \mu}\right) = A_2^{-1} w^{-2} \frac{\partial w}{\partial \mu} \quad ,$$

$$\frac{\partial}{\partial x} \left(\frac{\partial w}{\partial \mu}\right) = A_2^{-2} w^{-2} \{x + \mu(x^3 - x) + 8 \sin 3t\} \frac{\partial w}{\partial \mu} -$$

$$- A_2^{-2} w^{-1}(x^3 - x) - 24 A_2^{-2} w^{-1} \cos 3t \frac{\partial t}{\partial \mu} \quad ,$$

and by working much as in Lemma 1 we obtain as a first estimate that

$$\partial w / \partial \mu = \underline{O}(A_2^{-2}) \quad \text{and} \quad \partial t / \partial \mu = \underline{O}(A_2^{-3})$$

in $-1 \leqq x \leqq 1$. The approximate equations

$$\frac{\partial}{\partial x}\left(\frac{\partial t}{\partial \mu}\right) = A_2^{-1}\frac{\partial w}{\partial \mu} + \underline{0}\,(A_2^{-3})\,,$$

$$\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial \mu}\right) = -A_2^{-2}(x^3 - x) + \underline{0}\,(A_2^{-4})$$

can now be solved to give

$$\frac{\partial t}{\partial \mu} = -A_2^{-3}\left(\frac{x^5}{20} - \frac{x^3}{6} + \frac{x}{4} - \frac{2}{15}\right) + \underline{0}\,(A_2^{-5}),$$

which leads to (2.6) when we put $x = -1$.

Finally, we have

$$\frac{d(R^2)}{dx} = -\frac{2\mu(x^3 - x)\,(\,y - 3\cos 3t)}{y}$$

$$= -2\mu(x^3 - x) + 6\mu \cdot (x^3 - x)\cos 3t \cdot y^{-1}\,.$$

To find $R^2(\varphi_3) - R^2(\varphi_2)$ we must integrate this from $+1$ to -1 and only even powers of x will contribute to the integral. In the range we have

$$\cos 3t = \cos 3\varphi_2 + 3\sin 3\varphi_2 \cdot (x - 1) \cdot A_2^{-1} + \underline{0}\,(A_2^{-2})$$

and

$$y^{-1} = -A_2^{-1} + \underline{0}\,(A_2^{-3});$$

we see that

$$R^2(\varphi_3) - R^2(\varphi_2) = -6\mu \int_{+1}^{-1} (x^3 - x)x \cdot 3 \sin 3\varphi_2 \cdot A_2^{-2} dx + \underline{O}(|\mu|A_2^{-3})$$

$$= -\frac{24}{5} \mu \sin 3\varphi_2 \cdot A_2^{-2} + \underline{O}(|\mu|A_2^{-3}).$$

LEMMA 4. <u>Suppose</u> A_3 <u>is large and let</u> φ_4 <u>be the least</u> t <u>exceeding</u> φ_2 <u>such that the solution of</u> (1.2) <u>with</u> $x(\varphi_3) = -1$, $\dot{x}(\varphi_3) = -A_3$ <u>has</u> $x(t) = -1$. <u>Then</u> φ_4 <u>and</u> $\dot{x}(\varphi_4)$ <u>are analytic in</u> A_1^{-1} <u>and</u> φ_3,

$$\dot{x}(\varphi_4) = A_3 + \underline{O}(1)$$

<u>and</u>

$$\varphi_4 - \varphi_3 = \pi + \underline{O}(A_3^{-1}).$$

LEMMA 5. <u>Suppose</u> A_4 <u>is large and</u> $|\mu|$ <u>small, and let</u> $\varphi_5 = \varphi_5(A_4, \mu, \varphi_4)$ <u>be the least</u> t <u>exceeding</u> φ_4 <u>such that the solution of</u> (2.2) <u>with</u> $x(\varphi_4) = -1$, $\dot{x}(\varphi_4) = A_4$ <u>has</u> $x(t) = +1$. <u>Then</u> φ_5 <u>and</u> $\dot{x}(\varphi_5)$ <u>are</u> <u>analytic in</u> A_4^{-1}, μ <u>and</u> φ_5,

$$\dot{x}(\varphi_5) = A_4 + \underline{O}(A_4^{-1}),$$

$$\varphi_5(A_4, \mu, \varphi_4) = \varphi_5(A_4, 0, \varphi_4) + \frac{4}{15}\mu A_4^{-3} + \underline{O}(|\mu|A_4^{-5})$$

<u>and</u>

$$R^2(\varphi_5) - R^2(\varphi_4) = \frac{24}{5} \mu \sin 3\varphi_4 A_4^{-2} + \underline{O}(|\mu|A_4^{-3}). \tag{2.8}$$

Evidently we need only change x and t of Lemma 2 into x' and
t', and then write x = -x', t = t' + π .

LEMMA 6. _Suppose_ A_5 _is large and let_ φ_6 _be the least_ t
exceeding φ_5 _such that the solution of_ (1.2) _with_ $x(\varphi_5) = 1$, $\dot{x}(\varphi_5) = A_5$
has $\dot{x}(t) = 3$. _Then_ φ_6 _is analytic in_ A_5^{-1} _and_ φ_5.

3. Proof of Theorem 1

As was foreshadowed by the notation in the lemmas, we consider a
function which, in successive intervals of t, satisfies the equations
(2.1), (1.2), (2.2), (1.2), (2.2), (1.2): more precisely, we consider
x(t) with $x(0) = A_0$, $\dot{x}(0) = 3$ satisfying (2.1) in $0 \le t \le \varphi_1$, (1.2) in
$\varphi_1 \le t \le \varphi_2$ and so on. When we suppose A_0 large and use $A_1 = x(\varphi_1)$,
$-A_2 = \dot{x}(\varphi_2)$ etc., we combine the results of the lemmas, carrying λ and
μ as parameters. It is clear that, for all i, $A_i = A_0 + \underline{O}(1)$, and
indeed that this holds uniformly in λ, μ if, say, $|\lambda| \le 1$, $|\mu| \le 1$. We
suppose all error terms expressed in terms of A_0, and from now on drop the
subscripts.

For sufficiently small $|\lambda|$, $|\mu|$ and sufficiently large A, we see
that φ_6, $= \varphi_6(A, \lambda, \mu)$ say, is analytic in A^{-1}, λ, μ and can be expanded
as a multiple power series. The terms free of λ and μ have the sum
$\varphi_6(A, 0, 0)$ which must be precisely 2π since when $\lambda = \mu = 0$ the three
equations coalesce into (1.2), all of whose solutions have period 2π. Hence

$$\varphi_6(A, \lambda, \mu) = 2\pi - \frac{9}{2} \lambda A^{-2} + \frac{8}{15} \mu A^{-3} + \underline{O}(\lambda^2 + \mu^2) \tag{3.1}$$

and for sufficiently small $|\lambda|$ and $|\mu|$ we can solve the equation $\varphi_6(A, \lambda, \mu) = 2\pi$ to obtain λ as a series beginning

$$\lambda = \frac{16}{15} \mu A^{-1} + \dots \quad . \tag{3.2}$$

We write $\lambda(\mu, A)$ for the sum of this series; it is essential to observe that $\lambda(\mu, A)$ and μ have the same sign for small $|\mu|$ and large A.

It is clear that there exist $\mu*$ and $A*$, both positive, such that, for $0 < \mu \leq \mu*$ and $A \geq A*$, the series (3.2) is convergent and its sum positive and further that, when λ is taken equal to $\lambda(\mu, A)$,

(i) $R^2(\varphi_1) - R^2(0)$ for which we have the estimate (2.4), is positive,

(ii) $\sin 3\varphi_2 = \sin 3\pi/2 + \underline{O}(A^{-1})$ is negative and $\sin 3\varphi_4 = \sin 9\pi/2 + \underline{O}(A^{-1})$ is positive, and

(iii) $R^2(\varphi_3) - R^2(\varphi_2)$ and $R^2(\varphi_5) - R^2(\varphi_4)$, for which we have the estimates (2.7) and (2.8), are greater than $4\mu A^{-2}$. Corresponding to any μ in $0 < \mu \leq \mu*$ we define $g_\mu(x)$ by

$$g_\mu(x) = \begin{cases} x + \mu(x^3 - x) & \text{for } |x| \leq 1, \\ x & \text{for } |x| > 1. \end{cases}$$

Evidently $|g_\mu(x) - x| < \mu$ for all x.

We now proceed to define an $h_\mu(x, y)$. For $A > A*$, take $\lambda = \lambda(\mu, A)$ in (2.1) and consider the solution with $x(0) = A$, $\dot{x}(0) = 3$. If we write

$x(t, A)$ for this solution it is a consequence of our definitions that $\dot{x}(2\pi, A) = 3$. We now show that, as A varies (and λ with it, μ remaining fixed), the plane curves $x = x(t, A)$, $y = \dot{x}(t, A)$ cover a right-hand half-strip of $0 \leq y \leq 3$. In fact, when x and y are given we need to find large A and small t such that

$$x = A\cos t + \sin 3t + \tfrac{1}{2}\lambda(-At\cos t + \tfrac{3}{4}\cos 3t - \tfrac{3}{4}\cos t) + \ldots,$$

$$\text{(3.3)}$$

$$y = -A\sin t + 3\cos t + \tfrac{1}{2}\lambda(-A\cos t + At\sin t - \tfrac{9}{4}\sin 3t + \tfrac{3}{4}\sin t) + \ldots.$$

For our purposes we need only note that (3.3) imply

$$x = A + \underline{O}(1), \quad y = -At + 3 + \underline{O}(At^2) \tag{3.4}$$

and that we can throw (3.4) into the form

$$x^{-1} = A^{-1} + \underline{O}(A^{-2}), \quad (3 - y)x^{-1} = t + \underline{O}(t^2),$$

to which the standard implicit function theorem can at once be applied to give A^{-1} and t, at least for $x > x^* = x^*(\mu)$ say. We shall define $h_\mu(x,y)$ to have the constant value λ on any such curve lying <u>sufficiently far to the right</u> in the strip, that is, we shall choose $x^* = x^*(\mu) \geq x_1^*$ to satisfy a subsidiary condition and then, for each x and y with $x \geq x^*$ and $0 \leq y \leq 3$, determine A to satisfy (3.3) and write $h_\mu(x,y) = \lambda(\mu, A)$. For (x,y) outside this half-strip we define $h_\mu(x,y) = 0$. However x^* is chosen, we evidently have

$$h_\mu(x,y) \sim \frac{16}{135} \mu A^{-1} \sim \frac{16}{135} \mu x^{-1}$$

as $x \to \infty$, uniformly for $0 \leq y \leq 3$; our condition on $x*$ is that it should be so chosen that $h_\mu(x,y) < \mu$ for all x,y.

Given a positive ϵ, choose a $\mu < \min(\epsilon,\mu*)$, take h_μ and g_μ for f and g in (1.3), and consider the change of $R^2(t)$ for a solution with $x(0) = A > x*$ and $\dot{x}(0) = 3$. In the range 0 to φ_1, $R^2(t)$ increases, it is constant in the ranges φ_1 to φ_2, φ_3 to φ_4 and φ_5 to φ_6, and it increases, in each of φ_2 to φ_3 and φ_4 to φ_5, by $4\mu A^{-2}$ at least, whence

$$R^2(2\pi) - R^2(0) > 8\mu A^{-2} = 8\mu \{x(0)\}^{-2}.$$

Since $\dot{x}(2\pi) = 3$ we obtain similarly

$$R^2(4\pi) - R^2(2\pi) > 8\mu \{x(2\pi)\}^{-2},$$

and by repeating this we see that $R^2(t) \to \infty$.

Finally we must show that we can modify $h_\mu(x,y)$ to a continuous $f_\mu(x,y)$. If we write

$$c(y,\lambda) = \begin{cases} y, & 0 \leq y \leq \lambda, \\ \lambda, & \lambda \leq y \leq 3-\lambda, \\ 3-y, & 3-\lambda \leq y \leq 3, \end{cases}$$

then c is continuous for $0 \leq y \leq 3$, $0 \leq \lambda \leq 1$, and the equation

$$\ddot{x} + c(\dot{x},\lambda)\dot{x} + x = -8 \sin 3t \qquad\qquad (3.5)$$

is a modification of (2.1) which is identical with (2.1) in $\lambda \le y \le 3-\lambda$.
When A_o is large and λ small and positive, write $\varphi_{1c}(A_o,\lambda)$ for the
least positive t such that the solution of (3.5) with $x(0) = A_o$,
$\dot{x}(0) = 3$ has $\dot{x}(t) = 0$. Then just as for φ_1 in Lemma 1, φ_{1c} is analytic,
and with the accuracy given in (2.3) and (2.4) these estimates are valid
when φ_{1c} is written for φ_1. To verify this is tedious but straightforward,
and we may suppress the proof since it is clear on general grounds of contin-
uity that some modification of (2.1) must have these properties. Lemmas
2 and 6 are unaffected by the modification and we can combine the results of
our lemmas as above. If we modify our previous notations by inserting a sub-
script $\underline{c}$, we obtain the estimate (3.1) for $\varphi_{6c}(A, \lambda, \mu)$, then (3.2) for
$\lambda_c(\mu, A)$ and so on until we have been led to define $x_{1c}^* = x_{1c}^*(\mu)$.

Much as before, it is possible to choose $x_c^* \ge x_{1c}^*$ so that $h_{\mu c}(x,y)$
defined in the following manner should have $h_{\mu c}(x,y) < \mu$: given x and
y with $x \ge x_c^*$ and $0 \le y \le 3$ determine A to satisfy (3.3) and write

$$h_{\mu c}(x, y) = \begin{cases} y & \text{for } 0 \le y \le \lambda_c(\mu,A), \\ \lambda_c(\mu, A) & \text{for } \lambda_c(\mu, A) \le y \le 3 - \lambda_c(\mu, A), \\ 3 - y & \text{for } 3 - \lambda_c(\mu, A) \le y \le 3, \end{cases}$$

and $h_{\mu c}(x,y) = 0$ outside this half-strip. Evidently $h_{\mu c}(x, y)$ is continuous except when $x = x_c^*$, $0 \leq y \leq 3$. If we now write

$$f_\mu(x,y) = \begin{cases} (x - x_c^*)h_{\mu c}(x_c^* + 1, y) & \text{in } x_c^* \leq x \leq x_c^* + 1, 0 \leq y \leq 3, \\ \\ h_{\mu c}(x, y) & \text{elsewhere,} \end{cases}$$

we have a continuous function. Given a positive ϵ choose a $\mu < \min(\epsilon, \mu_c^*)$ and take f_μ and g_μ for f and g in (1.3). Our previous argument can now be repeated to show that, for any solution with $x*(0) = A > x_c^* + 1$ and $\dot{x}(0) = 3$, the corresponding $R^2(t)$ tends to infinity.

4. The Need to Introduce $g(x)$

In order to state a precise form of the assertion at the end of §1 we suppose $f(x, y)$ is a given function which

(i) <u>is continuous and non-negative inside</u> $x \geq x^* > 0, 0 \leq y \leq 3$, <u>and is</u> 0 <u>outside, and</u>

(ii) <u>is such that the solutions of</u>

$$\ddot{x} + f(x, \dot{x})\dot{x} + x = -8 \sin 3t \tag{4.1}$$

<u>are uniquely determined by initial values.</u>

When (i) and (ii) hold we shall say f <u>satisfies condition</u> C; with this terminology our result is

THEOREM 2. <u>If</u> f <u>satisfies condition</u> C <u>then, except in the case</u> <u>when</u> $f(x(t), \dot{x}(t)) = 0$ <u>for all</u> t, <u>a solution of</u> (4.1) <u>with</u> $x(0)$ <u>large and</u> $\dot{x}(0) = 3$ <u>has</u> $\dot{x}(2\pi) < 3$.

It will be observed that we have not prescribed that $f(x, y)$ is to be continuous on the boundaries of the half-strip. It will be clear that at the price of some complications in our enunciations and proofs we could also allow discontinuities inside the half-strip.

In a range of t in which $x(t)$, a solution of (4.1), is large we see that $y(t)$ is decreasing and that

$$\frac{dt}{dy} = - \frac{1}{x + f(x,\ y)y + 8 \sin 3t} \ , \quad \frac{dx}{dy} = y \ \frac{dt}{dy} \ . \qquad (4.2)$$

Here $t(y)$ is the function inverse to $y(t)$ and $x(y)$ is an abbreviation for $x(t(y))$; evidently when f satisfies condition C we can assert the uniqueness of solutions of (4.2). Similarly we can replace the comparison equation

$$x' + x' = -8 \sin 3t \qquad (4.3)$$

(in which, as throughout this paper, primes are used only as labels) by

$$\frac{dt'}{dy} = - \frac{1}{x' + 8 \sin 3t'} , \quad \frac{dx'}{dy} = y \ \frac{dt'}{dy} \ , \qquad (4.4)$$

where $t'(y)$ is inverse to $y'(t)$ and $x'(y)$ is written for $x'(t'(y))$.

LEMMA 7. <u>Suppose that</u> $f(x, y)$ <u>satisfies condition</u> C <u>and that</u> $(t(y), x(y))$ <u>is a solution of</u> (4.2) <u>and</u> $(t'(y), x'(y))$ <u>a solution of</u> (4.4) <u>such that</u>

(i) $x(y) > 8$ and $x'(y) > 8$ for $0 \leq y \leq 3$ and

(ii) $- \frac{1}{6} \pi < t(y) < \frac{1}{6} \pi$ and $- \frac{1}{6} \pi < t'(y) < \frac{1}{6} \pi$ for $0 \leq y \leq 3$.

If, further, there is an η with $0 \leq \eta < 3$ such that

$$t(\eta) \geq t'(\eta),$$

$$f(x(\eta), \eta) \, \eta \geq 0,$$

and
$$x(\eta) \geq x'(\eta)$$

with at least one inequality, then $t(y) > t'(y)$ and $x(y) > x'(y)$ for $\eta < y \leq 3$.

The hypothesis (i) guarantees that the denominators in

$$\frac{d(t-t')}{dy} = \frac{(x-x') + f(x, y)y + 8 (\sin 3t - \sin 3t')}{(x + f(x, y)y + 8 \sin 3t) (x' + 8 \sin 3t')}$$

$$\frac{d(x - x')}{dy} = y \cdot \frac{d(t - t')}{dy}$$

are positive, and the hypothesis (ii) that $\sin 3t > \sin 3t'$ when and only when $t > t'$. When $y = \eta$, the numerator of $d(t - t')/dy$ is positive; according as $\eta > 0$ or ~~$\geq$~~ $\eta = 0$ the numerator of $d(x - x')/dy$ is positive or zero. In either case there is an interval to the right of $y = \eta$ in which both $t(y) - t'(y)$ and $x(y) - x'(y)$ increase, and are therefore positive.

This interval must extend to $y = 3$ since otherwise we should have at least one of $t(y) - t'(y)$ and $x(y) - x'(y)$ attaining a maximum where its derivative was positive.

LEMMA 8. Suppose that $f(x,y)$ satisfies condition C, that B is large and that $0 < \beta < \frac{1}{6}\pi$. If $(t(y, \beta, B), \underline{x}(y, \beta, B))$ and $(t'(y, \beta, B), x'(y, \beta, B))$ are the solutions of (4.2) and (4.4) respectively with $t = t' = \beta$ and $x = x' = B$ where $y = 0$, then

$$t(3, \beta, B) \geq t'(3, \beta, B)$$

with   equality if and only if

$$f(x(y, \beta, B), y) = 0 \underline{\text{ for }} 0 \leq y \leq 3. \qquad (4.5)$$

If (4.5) holds, then $t(y, \beta, B) = t'(y, \beta, B)$ for $0 \leq y \leq 3$ and in particular for $y = 3$. If (4.5) does not hold there is a y_o with $0 < y_o < 3$ such that

$$f(x(y_o, \beta, B), y_o) > 0. \qquad (4.6)$$

Take $t(y, \beta, B)$ and $x(y, \beta, B)$ as the $t(y)$ and $x(y)$ of Lemma 7. If ϵ is small and positive take $t'(y, \beta, B - \epsilon)$ and $x'(y, \beta, B - \epsilon)$ as the $t'(y)$ and $x'(y)$ of Lemma 7. When B is large these four functions decrease slowly from their initial values and hence satisfy (i) and (ii) of Lemma 7. Since

$$t(0) = \beta = t'(0)$$

and

$$x(0) = B > B - \epsilon = x'(0),$$

all the hypotheses of Lemma 7 are satisfied with $\eta = 0$. Hence, for $0 < y \leq 3$,

$$t(y, \beta, B) > t'(y, \beta, B - \epsilon)$$

and

$$x(y, \beta, B) > x'(y, \beta, B - \epsilon).$$

Let ϵ tend to 0, and then write $y = y_o$; we obtain

$$t(y_o, \beta, B) \geq t'(y_o, \beta, B)$$

and

$$x(y_o, \beta, B) \geq x'(y_o, \beta, B).$$

Our result now follows from these and (4.6) by a further application of Lemma 7.